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Solving Forward-Looking Models of Cross-Country Adjustment within the Euro Area

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Abstract

This article introduces and applies two refinements to the algorithm of solving rational expectations models of a currency union. Firstly, building upon Klein (2000), it generalizes the standard methods of solving rational expectations models to the case of time-varying nonstochastic parameters, recurring in a finite cycle. Such a specification occurs in a simple stylized New Keynesian model of the euro area after a joint introduction of (i) rotation in the ECB Governing Council (as constituted by the Treaty of Nice) and (ii) home bias in the interest rate decisions preferred by its members. Secondly, we apply the method of Christiano (2002) to solve the model with heterogenous information sets. This is justified if we argue that the information set of domestic economic agents in a currency union is home-biased (i.e. foreign shocks enter only with a lag). Both methods of solution are illustrated with simulation results.

Keywords: solving rational expectations models, generalized Schur decomposition, heterogenous information sets, method of Christiano, divergences in EMU

JEL Classification: C32, C61, E52, F15.

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1 Introduction

The launch of the euro area project has opened space for empirical research and policy discussions on asymmetric shocks and adjustment mechanisms in their aftermath. Achieving a high capacity to absorb such shocks in the absence of autonomous monetary and exchange rate policy has become one of key economic policy targets, both for member and candidate countries. Market-based adjustment rests mainly upon the competitiveness channel (see European Commission 2006 and 2008). The adjustment process, however, may be hampered by the procyclical real interest rate mechanism, „Walters critique"; see Walters (1994).

Cyclical divergence and structural heterogeneity of the euro area countries were introduced into macroeconomic models in various ways. The main purpose of these extensions was to answer the question about the optimum monetary policy conduct in a monetary union. Following the works of Benigno (2004) and Benigno and Salido (2006), a number of authors investigated the welfare consequences of cross-country heterogeneity in intrinsic inflation persistence and market rigidities. Brissimis and Skotida (2008) consider regional differences in intertemporal elasticity of substitution, Lombardo (2006) - in price elasticity of imports, while Blessing (2008) assesses the welfare impact of structural heterogeneity in the presence of a nontradable sector. Further research includes differences in product and labour market flexibility (HM Treasury 2003, Rumler 2007), monetary transmission mechanisms; see Clausen and Hayo (2006) or business cycle synchronization.

Relatively less attention has been paid to the role of expectations in business cycle frictions of euro area member states, included in various dynamic stochastic general equilibrium models. The most widespread New Keynesian framework with nominal rigidities rests heavily upon forward-looking behaviour of economic agents. Consequently, rational expectations setup is normally applied for a fully specified system. Such system are solved using standard algorithms in the spirit of the seminal paper by Blanchard and Kahn (1980) and then used in simulation analyses.

Expecations play a key role in the functioning of both adjustment mechanisms in question. When a high inflation rate in an overheated economy translates into higher inflation expectations, then an asymmetric cyclical position - given weak (or no) reaction from the common central bank - results in a low real interest rate. This should additionally fuel economic activity, boost the cyclical amplitude and prolongue the period of adjustment. Nevertheless, if rational agents foresee that a protracted boom will undermine their country's external competitiveness, they anticipate the impact of deteriorated competitive position. Consequently, they should reduce their expectations of future output gap and inflation rate, which weakens the real interest rate mechanism.

This article aims to contribute to the existing literature on euro area modelling by including new features in the algorithm of solving out rational expectations. We argue that, under certain assumptions regarding monetary policy framework in the euro area and given the empirical evidence on expectations in the euro area from the

literature, the application of classical solution methods, as in Blanchard and Kahn (1980), can be insufficient.
Firstly, this is because the rotation scheme in the ECB Governing Council (henceforth: the Council), as constituted by the Treaty of Nice, may imply time-varying parameters in the Taylor rule approximating the ECB decisions. At present, the Council is composed of all the national central bank governors from the euro area countries with the right of vote in every decision meeting, as well as the ECB Board of Directors (henceforth: the Board). In this institutional setup, further euro area enlargement would imply a growing number of the Council members. This could lower the effectiveness of the decision process due to coordination problems, see e.g. Gerlach-Kristen (2005) and eventually motivated the introduction of a rotation system after the number of euro area members would exceed 15. This is the case since January 1st, 2009 when Slovakia adopted the euro. The system, however, was not yet implemented, as the reform can be postponed until there are 18 euro area member states. Under the Treaty, part of the governors would be rotationally excluded from the voting. A time-varying model is adequate when the rotation is coupled with some home bias of the Council members in preferred interest rate decisions.
Secondly, the inclination of economic agents to form inflation expectations first and above all on the basis of the events in the domestic economy justifies imposing heterogenous information sets across countries and hence across model equations. The simulation results presented in the paper suggest that both aspects can impact the volatility of inflation and output in the monetary union countries.
In contrast, the standard solution algorithm assumes both constant parameters and homogenous information sets for all equations. We address these issues individually by (i) generalizing the algorithm of Klein (2000) to the case of time-varying, nonstochastic parameters, (ii) directly applying the algorithm of Christiano (2002) to account for heterogeneity in information sets.
The rest of the paper is organized as follows. Section 2 develops a simple New Keynesian model of a monetary union and develops the extensions that challenge the standard solution procedure. Section 3 reviews the literature on solving rational expectations models and proposes a method of solving a model with variable coefficients is proposed. Section 4 presents an application of the considered methods in simulations. Section 5 concludes.

2 New Keynesian model of monetary policy and cross-country adjustment in the euro area

This Section presents the basic New Keynesian model, capturing the main features of monetary policy and competitiveness-based adjustment within a monetary union. Its main purpose is to motivate the modifications to the solution algorithm from an economic perspective.

2.1 The baseline setup

The model considered in this paper draws heavily on the workhorse 3-equation New Keynesian model for monetary policy analyses. It is composed of output gap equations (IS curves), inflation equations (Phillips curves) and nominal interest rate equation (central bank rule). The model has been extended to capture specific features of a group of open economies, forming a monetary union, with the competitiveness channel and the real interest rate effect.

The union-wide monetary policy is described by a Taylor (1993) rule with smoothing, see e.g. Sauer and Sturm (2003), for an extensive survey on Taylor rule applications as approximations to the ECB monetary policy:

$$i_t = (1-\rho)\left[r^* + \pi^* + \gamma_\pi\left(\pi_t - \pi^*\right) + \gamma_y y_t\right] + \rho i_{t-1} \tag{1}$$

with i_t - nominal central bank rate at time t, y_t - output gap of the monetary union, π_t - inflation rate in the monetary union, r^* - natural interest rate, π^* - inflation target of the common central bank, $\rho \in (0;1)$ - smoothing parameter, $\gamma_\pi > 1$, $\gamma_y > 0$ - parameters for central bank reaction to deviation of inflation from the inflation target and an open output gap respectively. The condition $\gamma_\pi > 1$ is required for the Taylor principle to be satisfied and the equilibrium to be determinate; see Taylor (1993). The inflation rate and output gap in the entire monetary union are calculated as weighted averages over the member countries:

$$\pi_t = \sum_{j=1}^{n} w_j \pi_{j,t} \tag{2}$$

$$y_t = \sum_{j=1}^{n} w_j y_{j,t} \tag{3}$$

Country weights (vector $\mathbf{w}_{n\times 1}$) reflect relative sizes of n economies ($j = 1, ..., n$) participating in the monetary union. Technically, as the ECB defines the price stability target in terms of the area-wide Harmonized Index of Consumer Prices dynamics („close to, but below 2%"), the weights could be associated with country weights for the area-wide HICP formula, published by the Eurostat. These are derived from national accounts as the share of consumption spendings of households in a given country in the analogous value for the euro area (see *Compendium of HICP reference documents* by Eurostat). They evolve sluggishly and most of their volatility was triggered by accessions to the euro area (Greece, Slovenia, Malta, Cyprus, Slovakia).

The inflation rate in country j (π_j) evolves according to a hybrid Phillips curve; see Galí and Gertler (1999) and Galí, Gertler, López-Salido (2001):

$$\pi_{j,t} = \omega_{f,j} E_t \pi_{j,t+1} + \omega_{b,j} \pi_{j,t-1} + \gamma_j y_{j,t} + \varepsilon^s_{j,t} \tag{4}$$

with $\varepsilon^s_{j,t}$ - cost-push shock in country j, $y_{j,t}$ - output gap in j. This specification results from maximizing future discounted flows of profits by firms. Only part of the producers are allowed to reset their prices in a given period; see Calvo (1983). They are split in two groups: those who reoptimize their price and those who perform a backward-looking indexation; see Galí and Gertler (1999) for an explicit derivation). Also, real marginal cost - the actual driving process for inflation - is here assumed to be a linear function of the output gap, which requires i.a. the assumption of perfect labour market flexibility.

The path of the output gap is determined by the following IS curve, augmented with open economy components; see Clarida, Galí, Gertler (2001) and Goodhart and Hofmann (2005):

$$\begin{gathered} y_{j,t} = \beta_{f,j} E_t y_{j,t+1} + \beta_{b,j} y_{j,t-1} - \beta_{r,j} \left(i_t - E_t \pi_{j,t+1} - r_j^* \right) + \\ - \beta_{c,j} \left(p_{j,t} - p_{-j,t} \right) + \beta_{s,j} y_{-j,t} + \varepsilon_{j,t}^d \end{gathered} \tag{5}$$

where $y_{-j,t}$ denotes the output gap outside j, $p_{j,t}$ – log-level of prices in j, $p_{-j,t}$ – log-level of prices outside j:

$$y_{-j,t} = \frac{\sum_{k,k \neq j} w_k y_k}{\sum_{k,k \neq j} w_k} \tag{6}$$

$$p_{j,t} = p_{j,t-1} + \pi_{j,t} \tag{7}$$

$$p_{-j,t} = \frac{\sum_{k,k \neq j} w_k p_{k,t-1}}{\sum_{k,k \neq j} w_k} + \frac{\sum_{k,k \neq j} w_k \cdot \pi_{k,t}}{\sum_{k,k \neq j} w_k} \tag{8}$$

The standard closed-economy specification has therefore been complemented with the real exchange rate divergence, $p_{j,t} - p_{-j,t}$, and external demand gap, $y_{-j,t}$. There is no nominal exchange rate dynamics between monetary union member countries, so the log-difference in price levels between the home and foreign economies can be interpreted as log-real exchange rate. This (or a very similar) specification of the IS curve has previously been considered i.a. by Kosior, Rozkrut, Toroj (2009), Torój (2009) and Menguy (2009).

The standard closed-economy IS curve, resulting from the Euler equation for consumption and market clearing conditions, is extended in correspondence with the point made by Clarida, Galí, Gertler (2001): demand conditions in a small open economy are determined by external demand conditions and the ratio of domestic prices (expressed in foreign currency) to the world's price level. Excess appreciation undermines the price competitiveness of domestic goods abroad ($\beta_c > 0$), and foreign economic downturns translate into slowdowns at home ($\beta_s > 0$). The rest of the parameters in (4) and (5), in line with the New Keynesian literature, should be positive.

The model composed of equations (1)-(8) can be written in the following matrix form:

$$\mathbf{A} E_t \mathbf{x}_{t+1} = \mathbf{B} \mathbf{x}_t + \mathbf{C} \mathbf{f}_t \tag{9}$$

whereby $\mathbf{x}_t$ contains all the model variables: $\mathbf{x}_t^T = \begin{bmatrix} 1 & \mathbf{P}_t^T & \mathbf{y}_{t-1}^T & \boldsymbol{\pi}_{t-1}^T & i_{t-1} & \mathbf{y}_t^T & \boldsymbol{\pi}_t^T \end{bmatrix}$, $\mathbf{f}_t$ – vector of random disturbances, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ – matrices of model parameters. The detailed description of matrices construction is explicitly provided in Appendix 1.

2.2 Extension 1: rotation scheme in the ECB Governing Council

In the baseline version of the model, the monetary policy conduct of a union's central bank was described by means of a Taylor rule operating on the union's aggregate inflation and output (1). Accordingly, in the euro area, the mandate of the ECB Governing Council is to maintain price stability in the entire union; see European Central Bank (2003). However, some opponents of the reform of voting modalities in the ECB argue that the new system could create incentives to abandon the pro-european perspective and renationalize the european monetary policy; see Belke (2003).
Namely, the new voting system in the ECB is claimed to additionally emphasize the national structure of the Council. This is because it introduces rotation into the Council, following similar solutions adopted e.g. in the USA. It was motivated by the euro area enlargements and coordination problems accompanying a decision process after the Council becomes of excess size; see Gerlach-Kristen (2005). Consequently, the euro area countries are to be split into 2 groups (when 16 to 21 states share the common currency) or 3 groups (for more than 21 states). With 2 groups, 5 greatest member states would delegate 4 central bank governors to the Council (in a rotating manner) and the rest of the member states - 11 representatives. With 3 groups, 5 biggest member states would delegate 4 governors, half of the states (rounded up for an odd number) - 8 governors and the rest - 3 governors; see Kosior, Rozkrut, Toroj (2009) for more technical details.
Consequences of the reform have been analyzed in the literature from different viewpoints and with various tools (see Table 1). On the normative level, the assessments of the new voting system vary. However, most of the authors arrive at the conclusion that a pro-european orientation of policymakers (i.e. zero home bias in interest rate decisions) maximizes the union's welfare after the reform. This contribution follows Kosior, Rozkrut, Torój (2009) in using the same macroeconomic framework, but extends that analysis by combining the rotation, home bias and a forward-looking perspective in a single model. Assume that every central bank governor implicitly prefers some nominal interest rate level, conditional upon the (possibly asymmetric) cyclical position of their country of origin:

$$i_{j,t} = (1-\rho)\left[r^* + \pi^* + \gamma_\pi\left(\pi_{j,t} - \pi_j^*\right) + \gamma_y y_{j,t}\right] + \rho i_{t-1} \qquad (10)$$

The literature differentiates between a cyclical and a structural (long-term) stress. If policymakers wanted to reduce the cyclical stress in their country of origin, see Calmfors (2007), Flaig and Wollmershäuser (2007), they would be inclined to vote in favour of interest rate changes towards $i_{j,t}$, even if these changes were at odds with (1). Calmfors (2007) and Flaig and Wollmershäuser (2007) also find some evidence that many euro area countries have suffered from the structural art of stress in the period 1999-2006. Nonetheless, the inclusion of structural stress into the governors' preferences would result in a „wandering" steady state of the model. In consequence, it is not impossible to analyze the second-moment properties of variables with the tools

Table 1: Voting reform in the ECB Governing Council: review of the literature

Study	Tools applied	Conclusions
Aksoy, de Grauwe, Dewachter (2002)	Standard New Keynesian framework. Cross-country heterogeneity stems from different monetary policy preferences, transmission mechanisms and business cycle developments.	The Board of Directors can effectively lead the monetary policy even when governors of individual central banks are home biased. Pro-european focus, however, maximizes the welfare.
Bénnasy-Quéré and Turkisch (2005)	Regional bias combined with rotation system. No endogeneity of future output or inflation with respect to interest rates.	Introduction of rotation will impact the effectiveness of ECB policy to a limited extent. Low rotation frequency would be beneficial to the „old" member states.
Paczyński (2006)	Regional bias combined with rotation system. Various degrees of home bias and decision rules considered.	Substantial home bias of the Council members might lead to serious policy errors.
Belke and Styczynska (2006)	Voting power indices and the regional bias of the Board's members.	The rotation system strengthens the ECB Board of Directors and - marginally - the big euro area economies. Sudden shifts in voting power could boost output and inflation volatility.
Fahrholz and Mohl (2006)	Voting power indices.	The rotation system strengthens the ECB Board of Directors and - marginally - the big euro area economies.
Kosior, Rozkrut, Torój (2009)	Voting power indices. New Keynesian model.	Pro-european focus of the Council's members minimizes output and inflation volatility.

Source: Kosior, Rozkrut, Torój (2009); author

applied in this paper, but this extension would make the results dependent on the estimates of long-term inflation differentials (Harrod-Balassa-Samuelson effect) and natural rates of interest. This is why we leave the structural aspect of the stress for future research. The final preference of the national central bank governor, declared in the voting, is defined as a weighted average of the „pro-european" rate in (1) and the preferred rate for his country of origin, as in (10):

$$\tilde{i}_{j,t} = (1-\alpha)\, i_t + \alpha i_{j,t} \tag{11}$$

The parameter $\alpha \in [0;1]$ measures the home bias in the decision of the Council's members. In this paper, we assume equal α across all Council members. With fully „pro-european" voters, $\alpha = 0$. The other limiting case of fully home-biased voters occurs when $\alpha = 1$.

The outcome of voting at t is approximated by the arithmetic average over preferences submitted by the governors allowed to vote at t. After the reform, only 15 (of a higher number of) national central bank governors would vote at a single meeting. Let $a_{j,t}$ be a dummy equal 1 when country j representative has got the right to vote at t and 0 otherwise. With these assumptions, the final interest rate decision of the ECB can be written as:

$$\bar{i}_t = \frac{1}{\sum_j a_{j,t}} \sum_{j=1}^{n} a_{j,t} \cdot \left((1-\alpha)\, i_t + \alpha i_{j,t} \right) \tag{12}$$

Also, this can easily be extended to incorporate a possible home bias of 6 Board's members. Assume that $a_{j,t} = 1$ if country j is at t represented either among the voting governors or in the Board (but not both), $a_{j,t} = 2$ if it is represented in both groups and $a_{j,t} = 0$ else. In this case, $\sum_j a_{j,t} = 21$. After some basic modifications, this setup allows us to set different degrees of home bias e.g. for the Board and the governors in the Council. In our numerical example, however, we do not consider this option.

Substituting (1)-(3) and (10) into (12), we obtain the final form of the Taylor rule for the ECB:

$$\begin{aligned} \bar{i}_t \quad = \quad & (1-\rho)\{r^* + \pi^* + \left[(1-\alpha)\,\mathbf{w}^T + \alpha \mathbf{a}_t^T\right] \gamma_\pi \left(\boldsymbol{\pi_t} - \boldsymbol{\pi^*}\right) + \\ & + \left[(1-\alpha)\,\mathbf{w}^T + \alpha \mathbf{a}_t^T\right] \gamma_y \mathbf{y_t}\} + \rho i_{t-1} \end{aligned} \tag{13}$$

where symbols in bold subscripted t are vectors of size $n \times 1$ containing a sequence of identically denoted variables over countries, and $\boldsymbol{\pi}^* = \pi^* \cdot \mathbf{1}_{n \times 1}$.

Note that if the home bias of the central bank governors is non-zero, the rotation scheme implies that the Taylor rule parameters for inflation rates and output gaps in individual economies vary in time. In other words, country weights in the nominal interest rate equation, as opposed to equations (2)-(3), are non-constant. In consequence, so is the matrix $\mathbf{B}$ in (9).

Non-constant parameters of the model (9) prevent us from applying standard solution methods.

2.3 Extension 2: country-specific information sets

Not only can the members of the Council exhibit home bias in their policy decisions. It can also be justified in the case of producers' and consumers' expectations. There is some empirical evidence in the literature in favour of such a bias.

The estimates of Taylor rule parameters for countries that formed the euro area in 1999 suggest that individual central banks conducted monetary policy in significantly different manners before they finally passed this responsibility to the ECB; see Eleftheriou, Gerdesmeier, Roffia (2006). Berger, Ehrmann, Fratzscher (2006) find econometric evidence that expectations of future ECB decisions substantially varied in the geografic dimension in the first years of the euro area. Woodford (2006) argues that the process of learning the new monetary policy regime among economic agents and hence altering their expectation formation habits might be protracted. The regime change clearly increased the degree of economic integration which not necessarily is accounted for in economic agents' information sets.

In a monetary union, agents could expect the foreign macroeconomic shocks to hit their domestic economy via a few channels. Firstly, the common central bank would react to foreign demand shocks with a move in the common policy rate, which would in turn translate directly into a change in domestic monetary conditions. Secondly, a foreign shock affects future price dynamics abroad. As a result, the real exchange rate would change – even when there were no direct price effects at home – which is another way to influence the domestic monetary conditions. Thirdly, foreign business cycle affects the domestic output due to international trade and investment links. Economic agents are therefore capable to predict an economic slowdown at home when they observe one in other countries.

Outside a monetary union, agents would have less incentive to monitor the foreign events. Firstly, the reaction of foreign central banks to foreign shocks does not automatically affect the nominal interest rates at home, which remain under the command of the domestic central bank. Secondly, a shock affecting foreign price dynamics does not necessarily translate into a shift in competitive position of domestic versus foreign producers, as measured with the real exchange rate. More precisely, this rate is also dependent on the nominal exchange rate, which can absorb asymmetric shocks; see Stążka (2009). Because of these two channels, Marzinotto (2008) argues e.g. that mid-size economies are largely at risk of excessive wage growth. Namely, their trade unions are too small to influence the ECB decisions in a significant way and too large to fear the loss of external competitiveness. Thirdly, the ample literature on the endogeneity of OCA criteria; e.g. integration of finance and trade in a common currency area, see e.g. Frankel and Rose (1998), or European Commission (2008) suggests growing interdependence of individual countries' output gaps after creating a monetary union.

For the reasons listed above, domestic agents - especially at the initial stages of participation in a monetary union - can be accustomed to form their expectations mainly on the basis of domestic events and to a lesser extent on the basis of foreign shocks. This would break the underlying assumptions of the standard expectations operator applied in the model (9), based on a common information set of agents in each country. As a consequence, this aspect of heterogeneity must be analyzed beyond the standard model solution methods.

More formally, in the baseline model, expectations of future output and inflation are formed in a rational manner, with all the information available at t being taken into account:

$$\begin{aligned} E\left(y_{j,t}\right) &= E\left(y_{j,t}|\Omega_t\right) \\ E\left(\pi_{j,t}\right) &= E\left(\pi_{j,t}|\Omega_t\right) \end{aligned} \tag{14}$$

with $\Omega_t = [\mathbf{f}_t, \mathbf{f}_{t-1}, ...]$ in equations (4)-(5). Should the information sets of agents in individual countries, however, be home biased, then country-specific output and inflation expectations would be defined as:

$$\begin{aligned} E(y_{j,t}) &= E(y_{j,t}|\Omega_{j,t}) \\ E(\pi_{j,t}) &= E(\pi_{j,t}|\Omega_{j,t}) \end{aligned} \tag{15}$$

whereby $\Omega_{j,t} \subseteq \Omega_t$ for each j.
The redefinition of the information sets does not affect the structural parameters of the forward-looking model (9). It affects, however, the algorithm that leads to model solution and hence changes the parameters of the solved model, impulse-response functions to shocks and second moment properties of variables. In this paper, we apply the method of Christiano (2002) designed to solve models with expectations defined as (15).

3 Solving linear rational expectations models

The solution of a dynamic linear model with rational expectations written as (9) is a transformation of (9) into a recursive law of motion; see Blanchard and Kahn (1980), Uhlig (1999), Klein (2000) and Sims (2001):

$$\mathbf{x}_t = \mathbf{M}\mathbf{x}_{t-1} + \mathbf{N}\mathbf{f}_t \tag{16}$$

This transformation is usually performed to run counterfactual simulations, impulse-response functions and second moment analysis; see DeJong and Dave (2007), Christiano (2002). Lindé (2005), Fuhrer and Rudebusch (2004) and other authors stress the possibility to use the mapping to estimate the parameters of the solved model (16) directly via full information maximum likelihood method. They show with Monte Carlo simulation exercises that such estimation outperforms the GMM estimator, traditionally applied in empirical investigations of the New Keynesian model.
Blanchard and Kahn (1980) solved the model (9) assuming nonsingularity of $\mathbf{A}$ and performing the Jordan decomposition on $\mathbf{A}^{-1}\mathbf{B}$. They also developed a milestone theorem about the existence and uniqueness of a solution: the number of variables predetermined at t included in $\mathbf{x}_t$ must equal the number of eigenvalues of the matrix $\mathbf{A}^{-1}\mathbf{B}$ that do not exceed 1 in absolute value (saddle-path stability).
The assumption of a nonsingular matrix $\mathbf{A}$ is relaxed by Klein (2000) as he applies generalized complex Schur decomposition to matrices $\mathbf{A}$ and $\mathbf{B}$. A similar method is proposed by Sims (2001) for the model (9) expressed in terms of true future values of variables and expectation errors rather than the expectation operator. The solution results from a unique linear mapping from $\mathbf{f}_t$ to the expectation errors. Söderlind (1999) applies a Klein-based algorithm and admits that generalized eigenvalues equal 1 in absolute terms can be classified as stable when the vector $\mathbf{x}_t$ contains variables that are explicitly nonstationary by construction (which also is the case in our model).

Uhlig (1999) proposes a method of undetermined coefficients, which reduces the problem to solving a matrix quadratic equation.

3.1 Standard solution

Without additional assumptions, developed in Subsections 2.2 and 2.3, we can apply a standard algorithm for solving the model (9) for simulations. In this Subsection, we present the method of Klein (2000), commonly applied in analyses based on dynamic stochastic general equilibrium setup. For future reference, we also indicate the points in the algorithm that will need to be changed when the assumption of parameter constancy will be relaxed.

Klein (2000) applies to matrices $\mathbf{A}$ and $\mathbf{B}$ in (9) a complex generalized Schur decomposition. It produces matrices $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{S}$ and $\mathbf{T}$ such that

$$\begin{aligned} \mathbf{QAZ} &= \mathbf{S} \\ \mathbf{QBZ} &= \mathbf{T} \end{aligned} \tag{17}$$

whereby $\mathbf{S}$ and $\mathbf{T}$ – upper triangular matrices, $\mathbf{Q}$ and $\mathbf{Z}$ – unitary matrices ($\mathbf{QQ}^H = \mathbf{Q}^H\mathbf{Q} = \mathbf{ZZ}^H = \mathbf{Z}^H\mathbf{Z} = \mathbf{I}$), where the superscript H denotes hermitian transpose ($\mathbf{S}$,$\mathbf{T}$,$\mathbf{Q}$ and $\mathbf{Z}$ are complex matrices). Without loss of generality, suppose that $\mathbf{x}_t$ is partitioned into $\mathbf{x}_{\mathbf{1},t}$ containing variables predetermined at t and $\mathbf{x}_{\mathbf{2},t}$ containing variables non-predetermined at t.

Klein (2000) defines the following substitution:

$$\tilde{\mathbf{x}}_t = \mathbf{Z}^H\mathbf{x}_t \tag{18}$$

Given (18) and after conformable partitioning of the matrices, we can express (9) as

$$\begin{bmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} \\ 0 & \mathbf{S_{22}} \end{bmatrix} E_t \begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t+1} \\ \tilde{\mathbf{x}}_{\mathbf{2},t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{T_{11}} & \mathbf{T_{12}} \\ 0 & \mathbf{T_{22}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t} \\ \tilde{\mathbf{x}}_{\mathbf{2},t} \end{bmatrix} + \begin{bmatrix} \mathbf{Q_1} \\ \mathbf{Q_2} \end{bmatrix} \mathbf{Cf}_t \tag{19}$$

Upper-triangularity of $\mathbf{S}$ and $\mathbf{T}$ has decoupled the lower portion of the redefined vector $\tilde{\mathbf{x}}_t$, which can be solved out of (19) as follows:

$$\tilde{\mathbf{x}}_{\mathbf{2},t} = \mathbf{T}_{\mathbf{22}}^{-1}\mathbf{S_{22}}E_t\tilde{\mathbf{x}}_{2,t+1} - \mathbf{T}_{\mathbf{22}}^{-1}\mathbf{Q_2}\mathbf{Cf}_t \tag{20}$$

Iterating (20) forward and using the law of iterated expectations, see Ljungqvist and Sargent (2004), we express $\tilde{\mathbf{x}}_{\mathbf{2},t}$ as the following infinite sum:

$$\tilde{\mathbf{x}}_{\mathbf{2},t} = \underset{k\to\infty}{lim} \left(\mathbf{T}_{\mathbf{22}}^{-1}\mathbf{S_{22}}\right)^k E_t\tilde{\mathbf{x}}_{2,t+1} - \mathbf{T}_{\mathbf{22}}^{-1} \left[\sum_{k=0}^{\infty} \left(\mathbf{S_{22}}\mathbf{T}_{\mathbf{22}}^{-1}\right)^k \mathbf{Q_2}\mathbf{C}E_t\mathbf{f}_{t+k}\right] \tag{21}$$

Note that the forward iteration of (20) to (21) exploits the assumption parameter constancy.

According to Proposition 1 by Blanchard and Kahn (1980), there exists a unique solution if the number of explosive eigenvalues (i.e. lying outside the unit circle) equals the number of non-predetermined variables. If this applied to the generalized eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, all of the eigenvalues concentrated in the block (2,2) would be explosive and hence the infinite sum would exist and the limit would converge to zero.
Another condition formulated by Blanchard and Kahn (1980) is that the exogenous variables in $\mathbf{f}_t$ do not „explode too fast" in expectations:

$$\forall t \quad \exists \bar{\mathbf{f}}_t \in R^k, \theta_t \in R \quad \forall i \geq 0 \quad -(1+i)^{\theta_t} \bar{\mathbf{f}}_t \leq E\left(f_{t+i}\right) \leq (1+i)^{\theta_t} \bar{\mathbf{f}}_t \tag{22}$$

In rational expectations models, autoregressive error terms are natural by construction and hence commonly applied, so let us assume a VAR representation; see e.g. Mavroeidis (2005):

$$\mathbf{f}_t = \mathbf{\Phi} \mathbf{f}_{t-1} + \boldsymbol{\varepsilon}_t \tag{23}$$

with $E_t \boldsymbol{\varepsilon}_{t+k} = 0$, $k = 1, 2, ...$. Stationarity of this process, i.e. nonexplosive eigenvalues of $\mathbf{\Phi}$, allow us to calculate (24). As $E_t \mathbf{f}_{t+k} = \mathbf{\Phi}^k f_t$ given (23), we can rewrite (24) as

$$\tilde{\mathbf{x}}_{\mathbf{2},t} = -\mathbf{T}_{\mathbf{22}}^{-1} \left[\sum_{k=0}^{\infty} \left(\underbrace{\mathbf{S}_{\mathbf{22}} \mathbf{T}_{\mathbf{22}}^{-1}}_{\mathbf{F}} \right)^k \underbrace{\mathbf{Q}_{\mathbf{2}} \mathbf{C}}_{\mathbf{G}} \underbrace{\mathbf{\Phi}}_{\mathbf{H}}{}^k \right] f_t = -\mathbf{T}_{\mathbf{22}}^{-1} \mathbf{L} \mathbf{f}_t \tag{24}$$

Following Klein (2000), we calculate the elements of $\mathbf{L}$ using the vectorization operator. First, premultiply $\mathbf{L}$ by $\mathbf{F}$ and postmultiply by $\mathbf{H}$ so that $\mathbf{FLH} = \sum_{i=0}^{+\infty} \mathbf{F}^{i+1} \mathbf{G} \mathbf{H}^{i+1} = \sum_{i=1}^{+\infty} \mathbf{F}^{i} \mathbf{G} \mathbf{H}^{i}$. Note that the only difference between this sum and $\mathbf{L}$ is the first component $\mathbf{G}$, so $\mathbf{L} - \mathbf{FLH} = \mathbf{G}$. Vectorize both sides and use the matrix identity $vec\left(\mathbf{ABC}\right) = \left(\mathbf{C}^T \otimes \mathbf{A}\right) vec\left(\mathbf{B}\right)$ to get $vec\left(\mathbf{L}\right) - \mathbf{H}^T \otimes \mathbf{F} \cdot vec\left(\mathbf{L}\right) = vec\left(\mathbf{G}\right)$. This can be premultiplied by $\left(\mathbf{I} - \mathbf{H}^T \otimes \mathbf{F}\right)^{-1}$, unless the matrix is singular. This yields:

$$vec\left(\mathbf{L}\right) = \left[\mathbf{I} - \mathbf{H}^T \otimes \mathbf{F}\right]^{-1} vec\left(\mathbf{G}\right) \tag{25}$$

We use the solution for the unstable component, (24), in the upper portion of (19):

$$\mathbf{S}_{\mathbf{11}} E_t \tilde{\mathbf{x}}_{\mathbf{1},t+1} - \mathbf{S}_{\mathbf{12}} \mathbf{T}_{\mathbf{22}}^{-1} \mathbf{L} \mathbf{\Phi} \mathbf{f}_t = \mathbf{T}_{\mathbf{11}} \tilde{\mathbf{x}}_{\mathbf{1},t} - \mathbf{T}_{\mathbf{12}} \mathbf{T}_{\mathbf{22}}^{-1} \mathbf{L} \mathbf{f}_t + \mathbf{Q}_{\mathbf{1}} \mathbf{C} \mathbf{f}_t \tag{26}$$

Rewriting (18) with the standard partitioning and using (24), we obtain::

$$\begin{bmatrix} \mathbf{x}_{\mathbf{1},t} \\ \mathbf{x}_{\mathbf{2},t} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{\mathbf{11}} & \mathbf{Z}_{\mathbf{12}} \\ \mathbf{Z}_{\mathbf{21}} & \mathbf{Z}_{\mathbf{22}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t} \\ -\mathbf{T}_{\mathbf{22}}^{-1} \mathbf{L} \mathbf{f}_t \end{bmatrix} \tag{27}$$

This allows us to express $\tilde{\mathbf{x}}_{\mathbf{1},t}$ in terms of $\mathbf{x}_{\mathbf{1},t}$:

$$\tilde{\mathbf{x}}_{\mathbf{1},t} = \mathbf{Z}_{\mathbf{11}}^{-1} \mathbf{x}_{\mathbf{1},t} + \mathbf{Z}_{\mathbf{11}}^{-1} \mathbf{Z}_{\mathbf{12}} \mathbf{T}_{\mathbf{22}}^{-1} \mathbf{L} \mathbf{f}_t \tag{28}$$

Using (28) in (29) and the predeterminacy of $\mathbf{x}_{\mathbf{1},t+1}$ at t, i.e. $\mathbf{x}_{\mathbf{1},t+1} = E_t\mathbf{x}_{\mathbf{1},t+1}$, we obtain:

$$\begin{aligned}\mathbf{x}_{\mathbf{1},t+1} \quad = \quad & \mathbf{Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}x}_{\mathbf{1},t} + [\mathbf{Z_{11}S_{11}^{-1}}\left(\mathbf{T_{11}Z_{11}^{-1}Z_{12}} - \mathbf{T_{12}}\right)\mathbf{T_{22}^{-1}L}- \\ & -\left(\mathbf{Z_{12}} - \mathbf{Z_{11}S_{11}^{-1}S_{12}}\right)\mathbf{T_{22}^{-1}L\Phi} + \mathbf{Z_{11}S_{11}^{-1}Q_1C}]\mathbf{f}_t\end{aligned} \tag{29}$$

Turning to $\mathbf{x}_{\mathbf{2},t}$, it can be expressed in terms of $\mathbf{x}_{\mathbf{1},t}$ and $\mathbf{f}_t$ using (27) and (28):

$$\mathbf{x}_{\mathbf{2},t} = \mathbf{Z_{21}Z_{11}^{-1}x}_{\mathbf{1},t} + \left(\mathbf{Z_{21}Z_{11}^{-1}Z_{12}} - \mathbf{Z_{22}}\right)\mathbf{T_{22}^{-1}Lf}_t \tag{30}$$

Equations (29) and (30) are the solution of the model (9).

3.2 Extension 1: model with time-varying parameters

The time-varing Taylor rule (13) requires solving a model like (9), but with time-varying parameters:

$$\mathbf{A}_{(t)}E_t\left(\mathbf{x}_{t+1}\right) = \mathbf{B}_{(t)}\mathbf{x}_t + \mathbf{C}_{(t)}\mathbf{f}_t \tag{31}$$

The solutions of models with variable coefficients proposed in the literature are usually designed for stochastic parameters (e.g. Markov switching with finite number of states). Farmer, Waggoner, Zha (2008) apply a minimum state variable solution that expands the vector x_t times the number of possible states and adjust the parameter matrices appropriately. The above system could also be treated as nonlinear in variables and solved numerically, given exogenous paths for $\mathbf{a}_t$ in (13). However, treating coefficients as nonstochastic corresponds better to the institutional setup (as long as it is credible for economic agents) and finding an analytical soultion to the problem seems to be more reliable and computationally efficient.

The solution proposed below builds upon the algorithm by Klein (2000), while introducing some necessary generalizations. In our model, it would suffice to allow for only one time-varying matrix – either $\mathbf{A}$ or $\mathbf{B}$, as with the assumptions from section 2.2 we could place all time-varying parameters into a single matrix. However, it does not really simplify further derivations as the generalized Schur decomposition with imposed eigenvalue ordering is unique and all the output matrices would inherit time-variability, no matter how many input matrices (1 or 2) would bear this feature. Neither does time-dependent $\mathbf{C}_{(t)}$ cause any significant analytical or numerical complication. This is why the system (31) and its solution is expressed in more general terms that our example would require.

It is useless to start with a single generalized Schur decomposition because the factor matrices we would obtain inherit the nonconstancy and parameter matrices for $\mathbf{x}_t$ and $E_t\left(\mathbf{x}_{t+1}\right)$ would not be upper triangular as we need. A time-varying matrix $\mathbf{Z}_{(t)}$ (see (17)) does not allow us to define the substitution (18) in a unique way. If we chose some arbitrary $\mathbf{Z}$ matrix in time (say, $\mathbf{Z}_{(t)}$), every equation would link two different variables, which would obviously leave no space to proceed. This problem would indeed be solved by finding the generalized Schur decomposition for $\mathbf{A}_{(t)}$ and

$\mathbf{B}_{(t+1)}$. However, in the latter case, there would be no $\mathbf{Q}_{(t)}$ to premultiply any equation leaving both matrices in question upper triangular (see (17)).

Instead, we exploit the assumption that $\mathbf{A}_{(t)}$ and $\mathbf{B}_{(t)}$ vary in time, but the values recur after m periods, i.e. $\mathbf{A}_{(t+j)} = \mathbf{A}_{(t+j+i\cdot m)}$ and $\mathbf{B}_{(t+j)} = \mathbf{B}_{(t+j+i\cdot m)}$ for each $j = 0, 1, ..., m-1$ and each $i \in \mathbb{N}$. Let us first factorize the matrices $\mathbf{A}_{(t)}$ and $\mathbf{B}_{(t)}$ for each period in the cycle using a sequence of generalized complex Schur decompositions:

$$\begin{aligned} \mathbf{Q}_{(t)}\mathbf{A}_{(t)}\mathbf{Z}_{(t)} = \mathbf{S}_{(t)} \\ \mathbf{Q}_{(t)}\mathbf{B}_{(t)}\mathbf{Z}_{(t)} = \mathbf{T}_{(t)} \end{aligned} \tag{32}$$

with $\mathbf{S}$,$\mathbf{T}$,$\mathbf{Q}$ and $\mathbf{Z}$ bearing the same properties as their counterparts in the standard case (Subsection 3.1). For the decomposition to be unique, we impose a restriction that diagonal elements of $\mathbf{S}$ and $\mathbf{T}$ are ordered in such a way that generalized eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ (equal $\frac{S_{i,i}}{T_{i,i}}$) ascend with rising index i.

Using (17) we can rewrite (31) for each t as:

$$\mathbf{S}_{(t)}\mathbf{Z}_{(t)}^{H}E_t\mathbf{x}_{t+1} = \mathbf{T}_{(t)}\mathbf{Z}_{(t)}^{H}\mathbf{x}_t + \mathbf{Q}_{(t)}\mathbf{C}_{(t)}\mathbf{f}_t \tag{33}$$

Let us write the equation for t, $t+1$, ..., $t+m-1$ and solve each of them for $\mathbf{x}$:

$$\begin{aligned} \mathbf{x}_t = \quad & \mathbf{Z}_{(t)}\mathbf{T}_{(t)}^{-1}\mathbf{S}_{(t)}\mathbf{Z}_{(t)}^{H}E_t\left(\mathbf{x}_{t+1}\right) - \mathbf{Z}_{(t)}\mathbf{T}_{(t)}^{-1}\mathbf{Q}_{(t)}\mathbf{C}_{(t)}\mathbf{f}_t \\ \mathbf{x}_{t+1} = \quad & \mathbf{Z}_{(t+1)}\mathbf{T}_{(t+1)}^{-1}\mathbf{S}_{(t+1)}\mathbf{Z}_{(t+1)}^{H}E_{t+1}\left(\mathbf{x}_{t+2}\right) - \mathbf{Z}_{(t+1)}\mathbf{T}_{(t+1)}^{-1}\mathbf{Q}_{(t+1)}\mathbf{C}_{(t+1)}\mathbf{f}_{t+1} \\ \vdots \quad & \\ \mathbf{x}_{t+m-1} = \quad & \mathbf{Z}_{(t+m-1)}\mathbf{T}_{(t+m-1)}^{-1}\mathbf{S}_{(t+m-1)}\mathbf{Z}_{(t+m-1)}^{H}E_{t+m-1}\left(\mathbf{x}_{t+m}\right) + \\ & -\mathbf{Z}_{(t+m-1)}\mathbf{T}_{(t+m-1)}^{-1}\mathbf{Q}_{(t+m-1)}\mathbf{C}_{(t+m-1)}\mathbf{f}_{t+m-1} \end{aligned} \tag{34}$$

A bottom-up sequence of substitutions and the law of iterated expectations allow us to write an equation for $\mathbf{x}_t$:

$$\begin{aligned} \mathbf{x}_t \quad = \quad & \underbrace{\left[\prod_{i=0}^{m-1}\mathbf{Z}_{(t+i)}\mathbf{T}_{(t+i)}^{-1}\mathbf{S}_{(t+i)}\mathbf{Z}_{(t+i)}^{H}\right]}_{\mathbf{D}_{(t)}}E_t\left(\mathbf{x}_{t+m}\right) + \\ & -\underbrace{\left\{\sum_{k=1}^{m-1}\left(\prod_{l=1}^{k}\mathbf{Z}_{(t+l-1)}\mathbf{T}_{(t+l-1)}^{-1}\mathbf{S}_{(t+l-1)}\mathbf{Z}_{(t+l-1)}^{H}\right)\mathbf{Z}_{(t+k)}\mathbf{T}_{(t+k)}^{-1}\mathbf{Q}_{(t+k)}\mathbf{C}_{(t+k)}E_t\mathbf{f}_{t+k} + \mathbf{Z}_{(t)}\mathbf{T}_{(t)}^{-1}\mathbf{Q}_{(t)}\mathbf{C}_{(t)}\mathbf{f}_t\right\}}_{\sum_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_t\mathbf{f}_{t+k}} \end{aligned} \tag{35}$$

Note that at this point we assume that economic agents trust in the new system's sustainability and know the sequence of countries' rotation.

Once again, we perform a complex generalized Schur decomposition of $\mathbf{D}_{(t)}$ and $\mathbf{I}$ (as the parameter matrix for $\mathbf{x}_t$):

$$\begin{aligned}\mathbf{Q}_{(t)}\mathbf{D}_{(t)}\mathbf{Z}_{(t)} = \mathbf{S}_{(t)} \\ \mathbf{Q}_{(t)}\mathbf{I}\mathbf{Z}_{(t)} = \mathbf{T}_{(t)}\end{aligned} \tag{36}$$

with the usual restriction on ordering generalized eigenvalues. Let us define an auxiliary variable:

$$\tilde{\mathbf{x}}_t = \mathbf{Z}^H_{(t)}\mathbf{x}_t \tag{37}$$

In line with the conventional treatment in the literature, let $\mathbf{x}_t$ be ordered in such a way that the first partition ($\mathbf{x}_{\mathbf{1},t}$) contains variables predetermined at t. Analogous partitioning of $\tilde{\mathbf{x}}_t$, substitution of (36) and (18) into (35), premultiplication by $\mathbf{Q}_{(t)}$ and conformable partitioning of $\mathbf{S}_{(t)}$, $\mathbf{T}_{(t)}$ and $\mathbf{Q}_{(t)}$ yield:

$$\begin{aligned}\begin{bmatrix} \mathbf{S}_{\mathbf{11}(t)} & \mathbf{S}_{\mathbf{12}(t)} \\ 0 & \mathbf{S}_{\mathbf{22}(t)} \end{bmatrix} E_t \begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t+m} \\ \tilde{\mathbf{x}}_{\mathbf{2},t+m} \end{bmatrix} &= \begin{bmatrix} \mathbf{T}_{\mathbf{11}(t)} & \mathbf{T}_{\mathbf{12}(t)} \\ 0 & \mathbf{T}_{\mathbf{22}(t)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_{\mathbf{1},t} \\ \tilde{\mathbf{x}}_{\mathbf{2},t} \end{bmatrix} + \\ &+ \begin{bmatrix} \mathbf{Q}_{\mathbf{1}(t)} \\ \mathbf{Q}_{\mathbf{2}(t)} \end{bmatrix} \left(\sum_{k=0}^{m-1} \mathbf{R}_{\mathbf{k}(t)} E_t \mathbf{f}_{t+k} \right)\end{aligned} \tag{38}$$

Following Klein (2000), we solve the lower, decoupled row of (19) for $\tilde{\mathbf{x}}_{2,t}$:

$$\tilde{\mathbf{x}}_{\mathbf{2},t} = \mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{S}_{\mathbf{22}(t)}E_t\tilde{\mathbf{x}}_{2(t),t+m} - \mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_t\mathbf{f}_{t+k}\right) \tag{39}$$

The finite cycle of length m, in which the parameters of $\mathbf{A}_{(t)}$ and $\mathbf{B}_{(t)}$ recur, implies $\mathbf{D}_{(t)} = \mathbf{D}_{(t+m)}$ and $\mathbf{R}_{\mathbf{k}(t)} = \mathbf{R}_{\mathbf{k}(t+m)}$ for each k. We can therefore shift (35) m periods forward without changing the parameters:

$$\mathbf{x}_{t+m} = \mathbf{D}_{(t)}E_{t+m}\left(\mathbf{x}_{t+2m}\right) + \Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_{t+m}\mathbf{f}_{t+m+k} \tag{40}$$

Matrices $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{S}$ and $\mathbf{T}$, resulting from the Schur decomposition of both matrices of interest in the above system, will equal those obtained in (36). Then, we can shift (20) by any multiple of m without changing the parameters:

$$\begin{aligned}\tilde{\mathbf{x}}_{2,t+m} &= \mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{S}_{\mathbf{22}(t)}E_{t+m}\tilde{\mathbf{x}}_{2,t+2m} - \mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_{t+m}\mathbf{f}_{t+m+k}\right) \\ \tilde{\mathbf{x}}_{2,t+2m} &= \mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{S}_{\mathbf{22}(t)}E_{t+2m}\tilde{\mathbf{x}}_{2,t+4m} - \mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_{t+2m}\mathbf{f}_{t+2m+k}\right) \\ &\vdots\end{aligned} \tag{41}$$

As in (34), a sequence of substitutions in (20) and (41) and iterating expectations allow us to express $\tilde{\mathbf{x}}_{\mathbf{2},t}$ as an infinite sum:

$$\tilde{\mathbf{x}}_{\mathbf{2},t} = -\sum_{i=0}^{+\infty}\left\{\left(\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{S}_{\mathbf{22}(t)}\right)^i\mathbf{T}^{-1}_{\mathbf{22}(t)}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}E_t\mathbf{f}_{t+i\cdot m+k}\right)\right\} \tag{42}$$

At this point, we need to know the expected path of future random disturbances, conditional on the information that agents have at t. Like in Subsection (3.1), we proceed with an autoregressive error term (23).
With $E_t\boldsymbol{\varepsilon}_{t+k}=0$, $k=1,2,...$, we can write the infinite sum (42) as

$$\begin{aligned}
\tilde{\mathbf{x}}_{\mathbf{2},t} &= -\sum_{i=0}^{+\infty}\left[\left(\mathbf{T}_{\mathbf{22}(t)}^{-1}\mathbf{S}_{\mathbf{22}(t)}\right)^{i}\mathbf{T}_{\mathbf{22}(t)}^{-1}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}\boldsymbol{\Phi}^{i\cdot m+k}\mathbf{f}_{t}\right)\right]= \\
&= -\sum_{i=0}^{+\infty}\left[\left(\underbrace{\mathbf{T}_{\mathbf{22}(t)}^{-1}\mathbf{S}_{\mathbf{22}(t)}}_{\mathbf{F}_{(t)}}\right)^{i}\underbrace{\mathbf{T}_{\mathbf{22}(t)}^{-1}\mathbf{Q}_{\mathbf{2}(t)}\left(\Sigma_{k=0}^{m-1}\mathbf{R}_{\mathbf{k}(t)}\boldsymbol{\Phi}^{k}\right)}_{\mathbf{G}_{(t)}}\cdot\left(\underbrace{\boldsymbol{\Phi}^{m}}_{\mathbf{H}_{(t)}}\right)^{i}\right]\mathbf{f}_{t}= \\
&= -\mathbf{L}_{(t)}f_{t}
\end{aligned} \tag{43}$$

Using (44) again, we calculate the elements of $\mathbf{L}_{(t)}$ by means of the vectorization operator:

$$vec\left(\mathbf{L}_{(t)}\right)=\left[\mathbf{I}-\mathbf{H}_{(t)}^{T}\otimes\mathbf{F}_{(t)}\right]^{-1}vec\left(\mathbf{G}_{(t)}\right) \tag{44}$$

The existence of the infinite sum stems from (i) fulfilled assumptions of the Blanchard-Kahn theorem (exactly all unstable generalized eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ concentrated in the partition (2,2) of matrices $\mathbf{S}$ and $\mathbf{T}$) as well as (ii) stability of the process (23) (eigenvalues of $\boldsymbol{\Phi}$ lower than 1 in absolute terms). Substitute (43) into (18) after premultiplication by $\mathbf{Z}_{(t)}$ and conformable partitioning:

$$\begin{bmatrix}\mathbf{x}_{\mathbf{1},t}\\ \mathbf{x}_{\mathbf{2},t}\end{bmatrix}=\begin{bmatrix}\mathbf{Z}_{\mathbf{11}(t)} & \mathbf{Z}_{\mathbf{12}(t)}\\ \mathbf{Z}_{\mathbf{21}(t)} & \mathbf{Z}_{\mathbf{22}(t)}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{x}}_{\mathbf{1},t}\\ -\mathbf{L}_{(t)}\mathbf{f}_{t}\end{bmatrix} \tag{45}$$

After solving out $\tilde{\mathbf{x}}_{\mathbf{1},t}$ from (27), we obtain a linear relationship linking $\mathbf{x}_{\mathbf{1},t}$, $\mathbf{x}_{\mathbf{2},t}$ and $\mathbf{f}_t$:

$$\mathbf{x}_{\mathbf{2},t}=\mathbf{Z}_{\mathbf{21}(t)}\mathbf{Z}_{\mathbf{11}(t)}^{-1}\mathbf{x}_{\mathbf{1},t}+\left(\mathbf{Z}_{\mathbf{21}(\mathbf{t})}\mathbf{Z}_{\mathbf{11}(t)}^{-1}\mathbf{Z}_{\mathbf{12}(t)}-\mathbf{Z}_{\mathbf{22}(t)}\right)\mathbf{L}_{(t)}\mathbf{f}_{t} \tag{46}$$

We exploit the predeterminacy of $\mathbf{x}_{\mathbf{1},t}$ to get:

$$\begin{aligned}
E_t\left(\mathbf{x}_{t+1}\right) &= E_t\begin{pmatrix}\mathbf{x}_{\mathbf{1},t+1}\\ \mathbf{x}_{\mathbf{2},t+1}\end{pmatrix}= \\
&=\begin{bmatrix}\mathbf{x}_{\mathbf{1},t+1}\\ \mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}_{\mathbf{11}(t+1)}^{-1}\mathbf{x}_{\mathbf{1},t+1}+\left(\mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}_{\mathbf{11}(t+1)}^{-1}\mathbf{Z}_{\mathbf{12}(t+1)}-\mathbf{Z}_{\mathbf{22}(t+1)}\right)\mathbf{L}_{(t+1)}\underbrace{E_t\mathbf{f}_{t+1}}_{\boldsymbol{\Phi}f_t}\end{bmatrix}= \\
&=\begin{bmatrix}\mathbf{I}\\ \mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}_{\mathbf{11}(t+1)}^{-1}\end{bmatrix}\mathbf{x}_{\mathbf{1},t+1}+\begin{bmatrix}0\\ \left(\mathbf{Z}_{\mathbf{21}(t+1)}\mathbf{Z}_{\mathbf{11}(t+1)}^{-1}\mathbf{Z}_{\mathbf{12}(t+1)}-\mathbf{Z}_{\mathbf{22}(t+1)}\right)\mathbf{L}_{(t+1)}\boldsymbol{\Phi}\end{bmatrix}\mathbf{f}_t
\end{aligned} \tag{47}$$

Using (46), we can also replace $\mathbf{x_{2,}}_t$ in $\mathbf{x}_t$:

$$\begin{aligned}\mathbf{x}_t &= \begin{bmatrix} \mathbf{x_{1,}}_t \\ \mathbf{x_{2,}}_t \end{bmatrix} = \begin{bmatrix} \mathbf{x_{1,}}_t \\ \mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{x_{1,}}_t + \left(\mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{Z_{12}}_{(t)} - \mathbf{Z_{22}}_{(t)}\right)\mathbf{L}_{(t)}\mathbf{f}_t \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)} \end{bmatrix}\mathbf{x_{1,}}_t + \begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{Z_{12}}_{(t)} - \mathbf{Z_{22}}_{(t)}\right)\mathbf{L}_{(t)} \end{bmatrix}\mathbf{f}_t\end{aligned} \tag{48}$$

In the example considered here, the vector of predetermined variables $\mathbf{x_{1,}}_t$ contains lags of all elements in $\mathbf{x_{2,}}_t$. Accordingly, some rows in $\mathbf{A}_{(t)}$, $\mathbf{B}_{(t)}$ and $\mathbf{C}_{(t)}$ were trivial identities defining the equivalence between some elements of $\mathbf{x_{1,}}_{t+1}$ and $\mathbf{x_{2,}}_t$. With the relation between $\mathbf{x_{1,}}_t$ and $\mathbf{x_{2,}}_t$ in hand, we can drop these rows and denote the remaining matrices as $\overline{\mathbf{A}}_{(t)}$, $\overline{\mathbf{B}}_{(t)}$ and $\overline{\mathbf{C}}_{(t)}$. Rewrite (31) without these rows, using (47) and (48):

$$\begin{aligned}&\overline{\mathbf{A}}_{(t)}\begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)} \end{bmatrix}\mathbf{x_{1,}}_{t+1} + \\ &\quad + \overline{\mathbf{A}}_{(t)}\begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\mathbf{Z_{12}}_{(t+1)} - \mathbf{Z_{22}}_{(t+1)}\right)\mathbf{L}_{(t+1)}\mathbf{\Phi} \end{bmatrix}\mathbf{f}_t = \\ &= \overline{\mathbf{B}}_{(t)}\begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)} \end{bmatrix}\mathbf{x_{1,}}_t + \overline{\mathbf{B}}_{(t)}\begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(\mathbf{t})}\mathbf{Z}^{-\mathbf{1}}_{\mathbf{11}(\mathbf{t})}\mathbf{Z_{12}}_{(\mathbf{t})} - \mathbf{Z_{22}}_{(\mathbf{t})}\right)\mathbf{L}_{(\mathbf{t})} \end{bmatrix}\mathbf{f}_t + \overline{\mathbf{C}}\mathbf{f}_t\end{aligned} \tag{49}$$

The solution of (49) with respect to $\mathbf{x}_{1,t+1}$ is the searched law of motion of the form (16):

$$\begin{aligned}\mathbf{x_{1,}}_{t+1} &= \left(\overline{\mathbf{A}}_{(t)}\begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)} \end{bmatrix}\right)^{-1}\overline{\mathbf{B}}_{(t)}\begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)} \end{bmatrix}\mathbf{x_{1,}}_t + \\ &+ \left(\overline{\mathbf{A}}_{(t)}\begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)} \end{bmatrix}\right)^{-1} \cdot \left(\overline{\mathbf{C}} + \overline{\mathbf{B}}_{(t)}\begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(t)}\mathbf{Z}^{-1}_{\mathbf{11}(t)}\mathbf{Z_{12}}_{(t)} - \mathbf{Z_{22}}_{(t)}\right)\mathbf{L}_{(t)} \end{bmatrix}\right)\mathbf{f}_t - \\ &- \left(\overline{\mathbf{A}}_{(t)}\begin{bmatrix} \mathbf{I} \\ \mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)} \end{bmatrix}\right)^{-1} \cdot \overline{\mathbf{A}}_{(t)}\begin{bmatrix} 0 \\ \left(\mathbf{Z_{21}}_{(t+1)}\mathbf{Z}^{-1}_{\mathbf{11}(t+1)}\mathbf{Z_{12}}_{(t+1)} - \mathbf{Z_{22}}_{(t+1)}\right)\mathbf{L}_{(t+1)}\mathbf{\Phi} \end{bmatrix}\mathbf{f}_t\end{aligned} \tag{50}$$

3.3 Extension 2: model with heterogenous information sets

We address the second extension considered in this article, i.e. the generalization of (14) to (15), by using the method of undetermined coefficients proposed by Christiano (2002). It is applicable when different groups of economic agents take their decisions with heterogenous knowledge of contemporaneous values of economic shocks. In our case, there are individual, heterogenous information sets of economic agents in individual countries of the monetary union. A detailed description of the method in full generality can be found in Christiano (2002). For the sake of presentational simplicity, this section consumes all the possible simplifications in the case that we are investigating.

Christiano (2002) ascribes an individual information set to every equation in the system and expectational terms in any equation are conditional upon the content of its own information set. Every information set contains all the past values of the random disturbances and part of its contemporaneous values (in an extreme case: all or none of them). He solves a linear dynamic rational expectations model of the form:

$$\varepsilon_t\left(\sum_{i=0}^{r}\boldsymbol{\alpha}_{\mathbf{i}}\mathbf{z}_{t+r-1-i}+\sum_{i=0}^{r-1}\boldsymbol{\beta}_{\mathbf{i}}\mathbf{s}_{t+r-1-i}\right)=0 \tag{51}$$

with $\mathbf{z}_t = \begin{bmatrix} \mathbf{z_{1}}_{,t} & \mathbf{z_{2}}_{,t} \end{bmatrix}^T$, $\mathbf{z_{1}}_{,t}$ - n_1-dimensional vector of endogenous variables non-predetermined at t, $\mathbf{z_{2}}_{,t}$ contains q lags of $\mathbf{z_{1}}_{,t}$ necessary to determine $\mathbf{z_{1}}_{,t+1}$ at $t+1$. In our model, the lag length does not exceed 1, which means $q = 0$, $r = 2$ and $\mathbf{z}_t = \mathbf{z}_{1,t} = \begin{bmatrix} 1 & \mathbf{P}_t & i_t & \mathbf{y}_t & \boldsymbol{\pi}_t \end{bmatrix}^T$ is a vector of length $\tilde{n} \equiv 2+3n$ (n – number of countries).

However, equation (51) is not equivalent to (2) due to a conceptual difference in expectation operators. When information sets for individual equations are heterogenous, $\varepsilon_t\,(.)$ denotes rational expectations based on an equation-specific (i.e. country-specific) information set:

$$\epsilon_t\left(\mathbf{x_{t+1}}\right) = \begin{bmatrix} E_t\left(\mathbf{x_{1}}_{,t+1}|\Omega_{1,t}\right) \\ E_t\left(\mathbf{x_{2}}_{,t+1}|\Omega_{2,t}\right) \\ \vdots \\ E_t\left(\mathbf{x_{n}}_{,t+1}|\Omega_{n,t}\right) \end{bmatrix} \tag{52}$$

At t, only the union's central bank is familiar with the entire vector of current country-specific demand and supply disturbances, $\mathbf{f}_t = \begin{bmatrix} \varepsilon_{y,1,t} & \dots & \varepsilon_{y,n,t} & \varepsilon_{\pi,1,t} & \dots & \varepsilon_{\pi,n,t} \end{bmatrix}^T$. The only contemporaneous values of shocks that economic agents in country j take into account are ones concerning their own country, i.e. $\varepsilon_{y,j,t}$ and $\varepsilon_{\pi,j,t}$. Shocks to the other economies enter the information set of country j agents with a one period lag.

The restrictions excluding some elements of $\mathbf{f}_t$ from some equations' information sets are summarized in the matrix $\boldsymbol{\tau}$ sized $2n \times \tilde{n}$. Its columns correspond with equations in the system, rows – with elements of $\mathbf{f}_t$; $\tau_{[i,j]} = 1$ when i-th innovation is included in the information set of equation j and $\tau_{[i,j]} = 0$ otherwise. In our setup,

$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{0}_{n\times 2n+1} & \mathbf{1}_{n\times 1} & \mathbf{I}_n & \mathbf{I}_n \\ \mathbf{0}_{n\times 2n+1} & \mathbf{1}_{n\times 1} & \mathbf{I}_n & \mathbf{I}_n \end{bmatrix}.$

In the framework of Christiano, we need to expand the random vector by its first lag: $\mathbf{s}_t = \begin{bmatrix} \mathbf{f}_t \\ \mathbf{f}_{t-1} \end{bmatrix}$ exactly due to the exclusion restrictions in τ. Like in Subsection 3.2, we assume a VAR respresentation (23) for $\mathbf{f}_t$. This method, however, additionally requires the knowledge of the variance-covariance matrix of $\boldsymbol{\varepsilon}$: $\mathbf{V_e} = E\left(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\right) = \begin{bmatrix} \mathbf{\Sigma_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma_s} \end{bmatrix}$ with independence of demand and supply disturbances assumed. Christiano (2002) emphasizes that this input is only required in the presence of at least one non-empty and incomplete information set. This implies the following VAR representation for $\mathbf{s}_t$:

$$\begin{bmatrix} \mathbf{f}_t \\ \mathbf{f}_{t-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{\Phi} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} \mathbf{f}_{t-1} \\ \mathbf{f}_{t-2} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \end{bmatrix} \tag{53}$$

Knowing the matrices $\boldsymbol{\alpha_0}, \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\beta_0}, \boldsymbol{\beta_1}$ and $\boldsymbol{\tau}$, $\mathbf{V_e}$, $\mathbf{P}$ (see Appendix 2), we can apply the method of Christiano (2002) to compute the matrices $\mathbf{M}, \mathbf{N}$ such that the solution to (51) is of the form

$$\mathbf{z}_t = \mathbf{M}\mathbf{z}_{t-1} + \mathbf{N}\mathbf{s}_t \tag{54}$$

Under complete information sets, the following framework is fully equivalent to the standard methods (Blanchard-Kahn, Klein, Sims or Uhlig). Moreover, $\mathbf{M}$ is always the same as derived via standard methods, i.e. independent on (in)completeness of information sets. When at least one of the information sets is incomplete, the key step in pinning down $\mathbf{N}$ is an orthogonal projection from the space of random disturbances included in an equation's information set (where $\mathbf{f}_{t-1}$ and part of $\mathbf{f}_t$ jointly belong) to the space of all contemporanous and lagged random disturbances (where $\mathbf{s}_t$ belongs). To obtain $\mathbf{M}$, merge $\mathbf{z_t}$ and $\mathbf{z}_{t-1}$ and write the system (51) skipping $\mathbf{s}_t$:

$$\underbrace{\begin{bmatrix} \boldsymbol{\alpha_0} & \mathbf{0}_{\tilde{n}\times\tilde{n}} \\ \mathbf{0}_{\tilde{n}\times\tilde{n}} & \mathbf{I_{\tilde{n}}} \end{bmatrix}}_{\mathbf{a}} \begin{bmatrix} \mathbf{z}_{t+1} \\ \mathbf{z}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ -\mathbf{I_{\tilde{n}}} & \mathbf{0}_{\tilde{n}\times\tilde{n}} \end{bmatrix}}_{\mathbf{b}} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \end{bmatrix} = \mathbf{0} \tag{55}$$

Factorize $\mathbf{a}$ and $\mathbf{b}$ using the generalized Schur decomposition (as in (17)). Arrange matrices $\mathbf{Q}, \mathbf{Z}, \mathbf{S}, \mathbf{T}$ so that the zeros on the main diagonal of $\mathbf{S}$ are located in its lower-right portion. This separates upper-left portions of $\mathbf{S}$ and $\mathbf{T}$ denoted $\mathbf{S_{11}}$ and $\mathbf{T_{11}}$ respectively. Partition conformably $\mathbf{Z}^H = \begin{bmatrix} \mathbf{Z_1^H} \\ \mathbf{Z_2^H} \end{bmatrix}$. Then, use the eigenvalue-eigenvector decomposition:

$$-\mathbf{S_{11}^{-1}}\mathbf{T_{11}} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1} \tag{56}$$

Let $\tilde{\mathbf{P}}$ denote the rows of $\mathbf{P}^{-1}$ corresponding to the eigenvalues exceeding 1 in absolute terms. Let $\mathbf{D} = \begin{bmatrix} \tilde{\mathbf{P}}\mathbf{Z_1}^H \\ \mathbf{Z_2}^H \end{bmatrix} = \begin{bmatrix} \underbrace{\mathbf{D_1}}_{\tilde{n}} & \underbrace{\mathbf{D_2}}_{\tilde{n}} \end{bmatrix}$. Matrix $\mathbf{M}$ is finally obtained as lower $\tilde{n}$ rows of the matrix $-\mathbf{D_1}^{-1}\mathbf{D_2}$.
To find $\mathbf{N}$, define for every equation i (i.e. for every column $\boldsymbol{\tau}_{[:,i]}$ in $\boldsymbol{\tau}$) matrix $\mathbf{R_i}$ as unity matrix in which the rows corresponding to zeros in the vector $\boldsymbol{\tau}_{[:,i]}$ were dropped. The orthogonal projection mentioned before implies matrices $\mathbf{C} = \sum_{i=0}^{\infty} \mathbf{\Phi}^i \mathbf{V_e} \left(\mathbf{\Phi}^T\right)^i$, $\varphi_i = \begin{bmatrix} \mathbf{R_i}\mathbf{C}\mathbf{R_i}^T & \mathbf{R_i}\mathbf{\Phi}\mathbf{C} \\ \mathbf{C}^T\mathbf{\Phi}^T\mathbf{R}_i^T & \mathbf{C} \end{bmatrix}$, $\boldsymbol{\phi}_i = \begin{bmatrix} \mathbf{C}\mathbf{R_i}^T & \mathbf{\Phi}\mathbf{C} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{a_i} & \mathbf{a_{i\theta}} \end{bmatrix} = \boldsymbol{\phi}_\mathbf{i}\boldsymbol{\varphi}_\mathbf{i}^{-1}$, whereby the number of columns in $\mathbf{a_i}$ equals the number of nonzero elements in $\boldsymbol{\tau}_{[:,i]}$.
Let

$$\mathbf{R} = \left[\begin{array}{cc|cc|c|cc} (\mathbf{a_1}\mathbf{R_1})^T & \mathbf{0} & \multicolumn{2}{c|}{\mathbf{0}} & \dots & \multicolumn{2}{c}{\mathbf{0}} \\ \mathbf{a}_{\mathbf{1}\boldsymbol{\theta}}^T & \mathbf{I} & & & & & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & (\mathbf{a_2}\mathbf{R_2})^T & \mathbf{0} & \dots & \multicolumn{2}{c}{\mathbf{0}} \\ & & \mathbf{a}_{\mathbf{2}\boldsymbol{\theta}}^T & \mathbf{I} & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c}{\vdots} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & \dots & (\mathbf{a_{\tilde{n}}}\mathbf{R_{\tilde{n}}})^T & \mathbf{0} \\ & & & & & \mathbf{a}_{\tilde{\mathbf{n}}\boldsymbol{\theta}}^T & \mathbf{I} \end{array}\right],$$

and $\tilde{\mathbf{R}}$ be defined as $\mathbf{R}$ with dropped zero rows resulting from zero elements in τ. Let $\tilde{d} = \tilde{R}vec\left[P^T\beta_0^T + \beta_1^T\right]$, and let $\tilde{q}$ be defined as matrix $\tilde{\mathbf{R}}\left(\boldsymbol{\alpha_0} \otimes \mathbf{P}^T + (\boldsymbol{\alpha_0}\mathbf{A} + \boldsymbol{\alpha_1}) \otimes \mathbf{I}\right)$ in which the columns were dropped whose numbers corresponded to the rows in $\tilde{R}$ that we had dropped before. The elements of $\mathbf{N}$ are defined by the relationship $vec\left(\mathbf{N}^T\right) = -\tilde{\mathbf{q}}^{-1}\tilde{\mathbf{d}}$, whereby the left-hand side vector - before de-vectorization into $\mathbf{N}^T$ - needs to be widened and filled with zeros at the indices of dropped rows in $\tilde{\mathbf{R}}$ (dropped columns in $\tilde{\mathbf{q}}$).

4 Simulation results

The models described in Section 2 and solved with the methods from Section 3 has beed used to simulate the path of the output gap and inflation rate of a generic monetary union's member countries with different assumptions regarding:

1. the home bias of the ECB Governing Council members (α);
2. the information set underlying the formation of expectations of future output and inflation in the countries of a monetary union.

Table 2 contains a set of model parameters used in the simulations. For simplicity, we assume homogenous parameters of the IS and Phillips curves across countries. Parameter values are median values of statistically significant estimates among 12 euro area countries over the time span 1999-2008, from Torój (2009). The parameters, following the dominant empirical approach in the New Keynesian literature, were estimated via generalized method of moments; see Galí and Gertler (1999), Galí, Gertler, López-Salido (2001), Goodhart and Hofmann (2005) with standard instrument sets for both curves. The parameters for the Taylor rule and AR processes of the random disturbances are parametrized as in the literature overview by Lindé (2005). Every pair of

Table 2: Parameters of the simulated model

ω_f	0.55	β_r	0.09	γ_π	1.5	ρ_π	0.1
ω_b	0.45	β_c	0.04	γ_y	0.5	ρ_y	0.5
β_f	0.5	β_s	0.09	ρ	0.5		
β_b	0.5	γ	0.05				
Source: Torój (2009)				Source: Source: Lindé (2005)			

variances compared below results from a path of variables generated with the same path of 10000 random shocks. Demand and supply disturbances were assumed to be independent. The variances of individual shocks were calibrated in such a way that the second moments of the baseline paths match those observed in the data on euro area countries inflation and output. The results generally confirm those obtained

Table 3: S.D. of output gap and inflation (expressed as a share of S.D. under baseline scenario)

	y	π
α	$w = 0.25$	$w = 0.25$
0	1.0000	1.0000
0.1	1.0003	1.0011
0.2	1.0008	1.0033
0.3	1.0017	1.0066
0.4	1.0028	1.0110
0.5	1.0042	1.0164

from a purely backward-looking model by Kosior, Rozkrut, Torój (2009), at least on the qualitative level. The rotation in the ECB Governing Council, coupled with some home bias in interest rate decisions among its members, can boost the variance of inflation and output gap. Table 3 presents the results of simulation when a monetary union consists of 4 equally sized countries and 2 country representatives participate in every vote. The cycle of rotation lasts 8 quarters, and there is a switch every 2 quarters. The standard deviations of output gap and inflation rise as α increases. For $\alpha = 0.5$, the standard deviation of the output gap is 0.42% higher than for $\alpha = 0$ (i.e. in the model with constant parameters). The standard deviation of inflation rises analogously by 1.64%.

When the country sizes differ, so do the results for big, mid-size and small economies. Table 4 presents the results of simulations generated with a 4-country model of monetary union with relative country sizes of 0.4, 0.3, 0.2 and 0.1. Once again, 2 country representatives vote at a time, the rotation cycle is 8 quarters long and the right of vote is granted to the governors 7, 5, 3 or 1 time a cycle, in line with their country size. It is only in the largest economy that the output gap volatility slightly declines

Table 4: S.D. of output gap and inflation (expressed as ratio to S.D. under baseline scenario); different country sizes

	y				π			
	$w =$				$w =$			
α	0.4	0.3	0.2	0.1	0.4	0.3	0.2	0.1
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.9985	0.9999	1.0012	1.0012	0.9961	1.0001	1.0031	1.0028
0.2	0.9971	0.9999	1.0026	1.0026	0.9927	1.0008	1.0066	1.0062
0.3	0.9958	1.0001	1.0041	1.0041	0.9901	1.0020	1.0107	1.0100
0.4	0.9947	1.0004	1.0058	1.0058	0.9881	1.0038	1.0152	1.0145
0.5	0.9937	1.0008	1.0076	1.0077	0.9867	1.0062	1.0201	1.0195

as α rises. The country of size 0.3 enjoys a slightly positive α for the same reason. However, starting from approximately $\alpha = 0.3$, it suffers from a rise in output volatility as the impact of „imported" instability from two small economies and the relative loss of the central bank's focus in favour of the greatest country begin to dominate. In the case of the smallest economies, this effect is visible for any positive home bias because their relative weight in the union-wide Taylor rule declines with growing α (it is more efficient to make big neighbours „pro-european" than to remain small and home-biased) and because they import each other's volatility simultaneaously. With $\alpha = 0.5$, the standard deviation of the output gap is ca. $0.76 - 0.77\%$ higher and inflation – ca. 2% higher than in the baseline scenario.

The impact of excluding the contemporaneous values of foreign demand and supply shocks from the information set of domestic agents is vague, even on the qualitative level. Table 5 is composed of standard deviations of output gaps when the information sets are incomplete, expressed as shares of standard deviations in the baseline scenario with complete information. Depending on the correlation of shocks between countries, serial correlation of country-specific shocks and – possibly – the country size, the incompleteness of the information set raises or reduces the volatility of output.

When the serial correlation of demand shocks is low, a home-biased information set reduces the variance of the output gap. Foreign demand shocks with low persistence have only limited impact on the domestic economy and start to influence domestic expectations once they have partially been absorbed. Note that the serial correlation of demand shocks at $0 - 0.2$ remains far lower than the empirical evidence would

suggest; see Lindé (2005).
Incomplete information sets also reduce the output volatility when the synchronicity of shocks between countries is high. In such a monetary union, the country-specific information set is sufficient to approximate some of the foreign noise and support the expectations as an auxiliary adjustment channel. When agents believe that shocks are correlated across countries, they do not fear that an asymmetric shock would induce inadequacy of the common interest rate and domestic macroeconomic aggregates.
On the other hand, under a high persistence and low symmetry of demand shocks, heterogenous and incomplete information sets yield higher variance of the output. Only with a lag does highly useful information arrive in agent's expectations. This hampers the „expectations" channel of adjustment and the stabilization of output around the potential level is more protracted, which generates a higher volatility. Note that such a stochastic environment is a contradiction of what the optimum currency area theory views as perfect (synchronized business cycles and at most temporary shocks).
Finally, note the effect in the row of Table 5 corresponding to empirically plausible values of demand shocks' serial correlation equal 0.6 and cross-country correlation equal 0.4. It suggests that in the mid-size economies, a limited information set might generate higher output volatility, whereas in big and small economies - lower volatility. Although this result seems to be quantitatively limited, the very fact that this model was capable to reproduce it might be seen as a weak confirmation of some tentative evidence reported in earlier literature. Namely, big economies benefit mainly from the stabilizing effects of common monetary policy and small ones - from the competitiveness channel. At the same time, expectation as a supportive channel of stabilization after asymmetric shocks could be particularly useful in mid-size member countries of a monetary union. In such countries, economic agents must therefore carefully watch the external environment. This weak implication of the model certainly needs further research in more complex DSGE models.

5 Conclusions

This paper generalizes the analytical methods of solving linear rational expectations models to the case of time-varying, nonstochastic parameters. This specification results from the inclusion of rotational voting system in the ECB when we combine it with home bias in voters' preferences. The assumption of a finite cycle in which the parameter values recur is thereby exploited. The solution is exemplified with the case of autoregressive random disturbances. The conditions for existence of a unique solution correspond in a straightforward way to the standard Blanchard-Kahn conditions.
We also apply the method of Christiano (2002) to introduce heterogeneity in individual countries' information sets. To the best of our knowledge, this algorithm has not been used before in modelling cross-country adjustment in the euro area. In our model, agents are home biased in their information sets in such a way that the foreign

Table 5: Heterogenous information sets – S.D. of output gap (expressed as ratio to S.D. under homogenous information sets)

cross-country correlation of demand shocks	serial correlation of demand shocks	country size 0.4	0.3	0.2	0.07	0.03
0	0	0.78	0.78	0.77	0.76	0.76
	0.2	0.91	0.91	0.91	0.90	0.90
	0.4	**1.13**	**1.13**	**1.13**	**1.14**	**1.14**
	0.6	**1.50**	**1.52**	**1.54**	**1.58**	**1.58**
	0.8	**2.22**	**2.30**	**2.44**	**2.52**	**2.57**
0.2	0	0.72	0.72	0.71	0.71	0.71
	0.2	0.85	0.85	0.84	0.84	0.84
	0.4	**1.05**	**1.05**	**1.06**	**1.05**	**1.05**
	0.6	**1.39**	**1.42**	**1.44**	**1.45**	**1.45**
	0.8	**2.02**	**2.11**	**2.20**	**2.30**	**2.32**
0.4	0	0.51	0.50	0.49	0.49	0.49
	0.2	0.60	0.59	0.59	0.58	0.57
	0.4	0.75	0.74	0.74	0.72	0.71
	0.6	0.99	0.99	**1.01**	**1.01**	0.99
	0.8	**1.48**	**1.55**	**1.55**	**1.59**	**1.64**
0.6	0	0.23	0.22	0.21	0.20	0.19
	0.2	0.28	0.26	0.26	0.24	0.23
	0.4	0.34	0.33	0.31	0.31	0.29
	0.6	0.47	0.45	0.43	0.42	0.40
	0.8	0.70	0.71	0.69	0.69	0.68
0.8	0	0.10	0.09	0.07	0.06	0.06
	0.2	0.11	0.10	0.08	0.07	0.06
	0.4	0.13	0.12	0.10	0.08	0.08
	0.6	0.16	0.14	0.13	0.10	0.10
	0.8	0.19	0.18	0.16	0.14	0.13

information arrives in their expectations with a one-period lag. This allows us to directly apply Christiano's algorithm.

The simulations based on the former solution method confirm the previous findings from the literature: the rotation in the ECB Governing Council, as implemented by the Treaty of Nice, coupled with home bias in interest rate decisions taken by the members of the Council, increases the volatility of output and inflation in most of the small and mid-size economies. However, the rise in standard deviation – with the parametrisation considered here – is very limited and amounts to a maximum of 2% in small economies.

Forming expectations on the country level with a partial, home-biased information set may in turn lead to a rise or a decline in the output volatility at home, depending on (i) the serial correlation of demand disturbances and (ii) their correlation across countries. When the properties of shocks do not conform to the optimum currency area theory, i.e. they are hardly synchronized and highly persistent, the output volatility can rise as a result of introducing incomplete information sets. This point clearly needs more investigation in the future.

A number of questions arise for further research, especially done with more sophisticated marcoeconomic models than the illustrative one proposed here. Fristly, it would be interesting to see how far the heterogeneity of information sets interacts with other aspects of euro area heterogeneity, such as market rigidities or inflation persistence. Secondly, the solution by Christiano (2002) seems to provide an underlying framework to derive an empirical test for heterogeneity of agent's information set in the economy of a real euro area member country. Thirdly, the solution with time-varying parameters could be applied in euro area DSGE models to verify the impact of the voting reform and Council members' home bias, especially in optimum policy models where the degree of home bias for individual countries would be a decision variable.

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References

[1] Aksoy Y., de Grauwe P., Dewachter H., (2002), Do Asymmetries Matter for European Monetary Policy?, *European Economic Review*, 46, 443–469.

[2] Belke A., (2003), The Rotation Model Is Not Sustainable, *Intereconomics*, 119–124.

[3] Belke A., Styczynska B., (2006), The Allocation of Power in the Enlarged ECB Governing Council: An Assessment of the ECB Rotation Model, *Journal of Common Market Studies*, 44, 865–897.

[4] Benigno P., (2004), Optimal monetary policy in a currency area, *Journal of International Economics*, 63.

[5] Benigno P., Salido J.D.L., (2006), Inflation persistence and optimal monetary policy in the euro area, *Journal of Money, Credit and Banking*, 38(3), 587–614.

[6] Berger H., Ehrmann M., Fratzscher M., (2006), Forecasting ECB Monetary Policy: Asccuracy Is (Still) a Matter of Geography, *IMF Working Paper*, 06/41.

[7] Blanchard O.J., Kahn C.M., (1980), The Solution of Linear Difference Models under Rational Expectations, *Econometrica*, 48(5), 1305–1311.

[8] Blessing J., (2008), How costly is lasting structural heterogeneity of Euro Area member states? Welfare results from a two-region two-sector DSGE model, working paper, *presented at:* 12$^{\text{th}}$ international ZEI Summer School in Bad Honnef, June 2008.

[9] Bénnasy-Quéré A., Turkisch E., (2005), ECB Governance in the Enlarged Eurozone, *cEPII Working Paper*, No 2005-20.

[10] Brissimis S.N., Skotida I., (2008), Optimal Monetary Policy in the Euro Area in the Presence of Heterogeneity, *Journal of International Money and Finance*, 27(2).

[11] Calmfors L. [ed.], (2007), *The EEAG report on the european economy 2007*, CESifo.

[12] Calvo G., (1983), Staggered prices in a utility-maximizing framework, *Journal of Monetary Economics*, 12(3).

[13] Christiano L.J., (2002), Solving Dynamic Equilibrium Models by a Method of Undetermined Coeffcients, *Computational Economics*, 20, 21–55.

[14] Clarida R., Galí J., Gertler M., (2001), Optimal Monetary Policy in Open versus Closed Economies: An Integrated Approach, *American Economic Review*, 91(2), 248–252.

[15] Clausen V., Hayo B., (2006), Asymmetric Monetary Policy Effects in EMU, *Applied Economics*, 38(10), 1123–1134.

[16] DeJong D.N., Dave C., (2007), *Structural Macroeconometrics*, Princeton University Press.

[17] Eleftheriou M., Gerdesmeier D., Roffia B., (2006), Monetary policy rules in the pre-EMU era. Is there a common rule?, *ECB Working Paper Series*, 659.

[18] European Central Bank, (2003), *Inflation Differentials in the Euro Area: Potential Causes and Policy Implications.*

[19] European Commission, (2006), Adjustment Dynamics in the Euro Area - Experiences and Challenges, *European Economy*, 6/2006.

[20] European Commission, (2008), EMU@10. Successes and Challenges after 10 Years of Economic and Monetary Union, *European Economy*, 2/2008.

[21] Fahrholz C., Mohl P., (2006), Does the EMU-enlargement Impair Price Stability in Europe? A Voting Power Analysis of the ECB Reform, *Homo Oeconomicus*, 23(2), 219–239.

[22] Farmer R.E., Waggoner D.F., Zha T., (2008), Minimal State Variable Solutions to Markov-Switching Rational Expectations Model, *Federal Reserve Bank of Atlanta Working Paper Series*, 2008-23.

[23] Flaig G., Wollmershäuser T., (2007), Does the Euro-Zone Diverge? A Stress Indicator for Analyzing Trends and Cycles in Real GDP and Inflation, *CESifo Working Paper*, 1937.

[24] Frankel J.A., Rose A.K., (1998), The Endogeneity of the Optimum Currency Area Criteria, *The Economic Journal*, 108, 1009–1025.

[25] Fuhrer J.C., Rudebusch G.D., (2004), Estimating the Euler equation for output, *Journal of Monetary Economics*, 51(6), 1133–1153.

[26] Galí J., Gertler M., (1999), Inflation dynamics: A structural econometric analysis, *Journal of Monetary Economics*, 44(2), 195–222.

[27] Galí J., Gertler M., López-Salido J.D., (2001), European Inflation Dynamics, *European Economic Review*, 45, 1237–1270.

[28] Gerlach-Kristen P., (2005), Monetary Policy Committees and Interest Rate Setting, *European Economic Review*, 50, 457–507.

[29] Goodhart C., Hofmann B., (2005), The Phillips Curve, the IS Curve and Monetary Transmission: Evidence for the US and the Euro Area, *CESifo Economic Studies*, 51(4).

[30] HM Treasury, (2003), UK Membership of the Single Currency. An Assessment of the Five Economic Tests, *available at:* `http://www.hm-treasury.gov.uk/`.

[31] Klein P., (2000), Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model, *Journal of Economic Dynamics and Control*, 24, 1405–1423.

[32] Kosior A., Rozkrut M., Torój A., (2009), Rotation Scheme of the ECB Governing Council: Monetary Policy Effectiveness and Voting Power Analysis, [in:] *Raport nt. pełnego uczestnictwa Rzeczypospolitej Polskiej w trzecim etapie Unii Gospodaczej i Walutowej*. Projekty badawcze, *available at:* `http://www.nbp.pl/`, 9, 53–101.

[33] Lindé J., (2005), Estimating New-Keynesian Phillips Curves: A Full Information Maximum Likelihood Approach, *Journal of Monetary Economics*, 52, 1135–1149.

[34] Ljungqvist L., Sargent T.J., (2004), *Recursive Macroeconomic Theory*, MIT Press.

[35] Lombardo G., (2006), Inflation targeting rules and welfare in an asymmetric currency area, *Journal of International Economics*.

[36] Marzinotto B., (2008), Why so much wage restraint in EMU? The role of country size, *University of Teramo Department of Communication Working Paper*, 35.

[37] Mavroeidis S., (2005), Identifcation Issues in Forward-Looking Models Estimated by GMM, with an Application to the Phillips Curve, *Journal of Money, Credit and Banking*.

[38] Menguy S., (2009), Heterogeneity in inflation persistence and monetary policy in a monetary union, *presented at:* International Conference "10 Years Of The Euro: Adjustment In Capital And Labour Markets", May 7-8, 2009, Braga (Portugalia).

[39] Paczyński W., (2006), Regional Biases and Monetary Policy in the EMU, *working paper*.

[40] Rumler F., (2007), Estimates of the Open Economy New Keynesian Phillips Curve for Euro Area Countries, *Open Economies Review*, 18.

[41] Sauer S., Sturm J.E., (2003), Using Taylor Rules to Understand ECB Monetary Policy, *German Economic Review*, 8(3), 375–398.

[42] Sims C.A., (2001), Solving Linear Rational Expectations Models, *Computational Economics*, 20, 1–20.

[43] Söderlind P., (1999), Solution and estimation of RE macromodels with optimal policy, European Economic Review, 43, 813–823.

[44] Stążka A., (2009), The Flexible Exchange Rate as a Stabilising Instrument: The Case of Poland, [in:] *Raport nt. pełnego uczestnictwa Rzeczypospolitej Polskiej w trzecim etapie Unii Gospodaczej i Walutowej*. Projekty badawcze, *available at:* `http://www.nbp.pl/`.

[45] Taylor J.B., (1993), Discretion versus policy rules in practice, [in:] *Carnegie-Rochester Conference Series on Public Policy*, 39(1), 195–214.

[46] Torój A., (2009), Macroeconomic adjustment and heterogeneity in the euro area, *Materiały i Studia NBP*, 54.

[47] Uhlig H., (1999), A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily, chapter, [in:] *Computational Methods for the Study of Dynamic Economies*, Oxford Universtiy Press, 30–61.

[48] Walters A., (1994), Walters Critique, chapter, [in:] *The Economics and Politics of Money: The Selected Essays of Alan Walters*, U.K. Elgar.

[49] Woodford M., (2006), Interpreting Inflation Persistence: Comments on the Conference on "Quantitative Evidence on Price Determination", *Journal of Money, Credit and Banking*, 39(s1), 203–210.

Appendix 1: Construction of matrices A, B and C in Subsection 3.2

$$\mathbf{A} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \boldsymbol{c_y} & -\boldsymbol{\beta_c} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & -\boldsymbol{\beta}_{\mathbf{r}} & \boldsymbol{\beta_f} & \boldsymbol{\beta_r} \\ \boldsymbol{c_\pi} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \boldsymbol{\omega_f} \end{bmatrix},$$

$$\mathbf{B_{(t)}}^{\mathrm{T}} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & t_1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & -\boldsymbol{\beta_b} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & -\boldsymbol{\omega_b} \\ 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \rho & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times n} & 0.25\cdot\mathbf{I}_n & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{t}_{2(t)} & \boldsymbol{\beta}_{\mathbf{s}} & -\gamma \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{t}_{3(t)} & \mathbf{0}_{n\times n} & \mathbf{I}_n \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{0}_{3n+1,2n} \\ \boldsymbol{I}_{2n} \end{bmatrix}, \ \mathbf{c}_{\boldsymbol{\pi}} = \begin{bmatrix} (1-\omega_{b,1}-\omega_{f,1})\,\pi_1^* \\ (1-\omega_{b,2}-\omega_{f,2})\,\pi_2^* \\ \vdots \\ (1-\omega_{b,27}-\omega_{f,27})\,\pi_n^* \end{bmatrix},$$

$$\mathbf{c}_{\mathbf{y}} = \begin{bmatrix} \beta_{r,1}r_1^* \\ \beta_{r,2}r_2^* \\ \vdots \\ \beta_{r,27}r_n^* \end{bmatrix},$$

$$\boldsymbol{\gamma} = diag\begin{bmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_n \end{bmatrix}, \boldsymbol{\omega}_{\mathbf{f}} = diag\begin{bmatrix} \omega_{f,1} & \omega_{f,2} & \dots & \omega_{f,n} \end{bmatrix},$$

$$\boldsymbol{\omega}_{\mathbf{b}} = \begin{bmatrix} \omega_{b,1} & \omega_{b,2} & \dots & \omega_{b,n} \end{bmatrix},$$

$$\boldsymbol{\beta}_{\mathbf{f}} = diag\begin{bmatrix} \beta_{f,1} & \beta_{f,2} & \dots & \beta_{f,n} \end{bmatrix},$$

$$\boldsymbol{\beta}_{\mathbf{b}} = diag\begin{bmatrix} \beta_{b,1} & \beta_{b,2} & \dots & \beta_{b,n} \end{bmatrix},$$

$$\boldsymbol{\beta}_{\mathbf{r}} = diag\begin{bmatrix} \beta_{r,1} & \beta_{r,2} & \dots & \beta_{r,n} \end{bmatrix},$$

$$\boldsymbol{\beta_s} = \begin{bmatrix} 1 & -\beta_{s,1}\frac{w_2}{1-w_1} & ... & -\beta_{s,1}\frac{w_n}{1-w_1} \\ -\beta_{s,2}\frac{w_1}{1-w_2} & 1 & ... & -\beta_{s,2}\frac{w_n}{1-w_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{s,n}\frac{w_1}{1-w_n} & -\beta_{s,n}\frac{w_2}{1-w_n} & ... & 1 \end{bmatrix},$$

$$\boldsymbol{\beta_c} = \begin{bmatrix} \beta_{c,1} & -\beta_{c,1}\frac{w_2}{1-w_1} & ... & -\beta_{c,1}\frac{w_n}{1-w_1} \\ -\beta_{c,2}\frac{w_1}{1-w_2} & \beta_{c,2} & ... & -\beta_{c,2}\frac{w_n}{1-w_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{c,n}\frac{w_1}{1-w_n} & -\beta_{c,n}\frac{w_2}{1-w_n} & ... & \beta_{c,n} \end{bmatrix},$$

$t_1 = (1-\rho)(r^* + \pi^* - \gamma_\pi \pi^*)$,
$\mathbf{t}_{2(t)} = (1-\rho)\gamma_y\left((1-\alpha)\mathbf{w}^T + \alpha \boldsymbol{a}_t^T\right)$, $\mathbf{t}_{3(t)} = (1-\rho)\gamma_\pi\left((1-\alpha)\mathbf{w}^T + \alpha \boldsymbol{a}_t^T\right)$.

Appendix 2: Construction of matrices used in Subsection 3.3

$$\boldsymbol{\alpha_0} = \begin{bmatrix} \mathbf{0}_{n+2\times n+2} & \mathbf{0}_{n+2\times n} & \mathbf{0}_{n+2\times n} \\ \mathbf{0}_{n\times n+2} & \boldsymbol{\beta_f} & \boldsymbol{\beta_r} \\ \mathbf{0}_{n\times n+2} & \mathbf{0}_{n\times n} & \boldsymbol{\omega_f} \end{bmatrix},$$

$$\boldsymbol{\alpha_2} = \begin{bmatrix} -1 & \mathbf{0}_{1\times n} & 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{I}_n & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ 0 & \mathbf{0}_{1\times n} & \rho & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \boldsymbol{\beta_b} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \boldsymbol{\omega_b} \end{bmatrix},$$

$$\boldsymbol{\alpha_1} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n} & 0 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & -\mathbf{I}_n & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & 0,25\mathbf{I}_n \\ (1-\rho)(r^* + \pi^* - \gamma_\pi \pi^*) & \mathbf{0}_{1\times n} & -1 & (1-\rho)\gamma_y \mathbf{w}^T & (1-\rho)\gamma_\pi \mathbf{w}^T \\ \mathbf{c_y} & -\boldsymbol{\beta_c} & -\boldsymbol{\beta_r} & -\boldsymbol{\beta_s} & \mathbf{0}_{n\times n} \\ \mathbf{c_\pi} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \boldsymbol{\gamma} & -\mathbf{I}_n \end{bmatrix},$$

$$\boldsymbol{\beta_0} = \mathbf{0}_{\tilde{n}\times 4n},\ \boldsymbol{\beta_1} = \begin{bmatrix} \mathbf{0}_{(n+1)\times n} & \mathbf{0}_{(n+1)\times n} & \mathbf{0}_{(n+1)\times 2n} \\ \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 2n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times 2n} \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times 2n} \end{bmatrix}.$$